Subspace Techniques of Acoustic Source Detection and Localization in Shallow ocean

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OVERVIEW

1. Models of signal and noise vectors
2. Definitions of modal subspace and signal subspace
3. Passive detection of targets
4. Bearing estimation
SIGNAL MODEL

Source-receiver geometry

\[ p(r, z_s, \theta, z) = b \sum_{m=1}^{M} \psi_m(z_s) \psi_m(z) \frac{e^{ik_mr}}{\sqrt{k_m r}} \]

\( \psi_m(z) \): Discrete normal mode functions
\( k_m \): Modal wavenumbers
Vertical Linear Array

Acoustic pressure signal vector $\mathbf{p}$ in frequency domain at a vertical linear array (VLA) of $N$ scalar sensors:

$$
\mathbf{p} = \mathbf{H}_v \mathbf{q}
$$

$$
\mathbf{H}_v = \begin{pmatrix}
\psi_1(z_1) & \cdots & \psi_M(z_1) \\
\vdots & \ddots & \vdots \\
\psi_1(z_N) & \cdots & \psi_M(z_N)
\end{pmatrix}
$$

$z_1, \ldots, z_N$ are sensor depths.

$\mathbf{q} = [q_1, \ldots, q_M]^T$ is the vector of complex mode amplitudes

$$
q_m = bg_m(r) \psi_m(z_s)
$$
Horizontal Linear Array

Array signal vector for a uniform horizontal linear array (HLA) of $N$ scalar sensors at depth $z_r$ with inter-sensor distance $d$

$$p = H_h q$$

where

$$H_h = 
\begin{pmatrix}
\psi_1(z_r) & \cdots & \psi_M(z_r) \\
\vdots & \ddots & \vdots \\
\psi_1(z_r) e^{i(N-1)dk_1 \cos \theta} & \cdots & \psi_M(z_r) e^{i(N-1)dk_M \cos \theta}
\end{pmatrix}$$
DATA MODEL

• Signal is corrupted by additive white Gaussian noise

\[ x = Hq + w \]

\( x, w : N \times 1 \text{ vectors} \)

\( H = H_v \text{ or } H_h : N \times M \text{ matrix} \)

\( R_w = \sigma^2 I : \text{Noise covariance matrix of size } N \times N \)
MODAL SUBSPACE

\[ x = s + w, \quad s = Hq \]

Let \( N > M \). Columns of \( H \) are linearly independent. Modal subspace \( M = \text{span of the columns of } H \). \( M \) is \( M \)-dimensional subspace of \( \mathbb{C}^N \), \( N > M \).

Vertical array: \( M \) is independent of \( r, z, \theta \).

Horizontal array: \( M = M(\theta) \) is independent of \( r, z \).
CFAR DETECTION OF UNKNOWN SIGNAL

\begin{align*}
H_0 : \quad & y = n \\
H_1 : \quad & y = y_s + n
\end{align*}

- Noise \( n \) is a zero-mean complex Gaussian random \( N \)-vector, with unknown covariance matrix \( R_n \).
- Signal \( y_s \) is an unknown deterministic complex \( N \)-vector.

\( y \) : primary data vector.
\( y(1), y(2), \ldots, y(K) \) : secondary data vectors, which are mutually independent and have same statistical properties as those of noise vector \( n \) when the signal is present.

\( K \geq N \)
... CFAR Detection of Unknown Signal

Generalized Likelihood Ratio:

\[
\text{GLR} = \frac{\text{Maximized likelihood function under } H_1}{\text{Maximized likelihood function under } H_0} = y^H S^{-1} y + 1
\]

where \( S \) is the sample correlation matrix

\[
S = \sum_{j=1}^{K} y(j)y^H(j)
\]

GLRT: \( \gamma(y, S) \equiv y^H S^{-1} y \geq (t_0 - 1) \)

\( t_0 \) is preset threshold
Probability of false alarm:

\[
P_{FA} = \frac{K!}{(K-N)!(N-1)!} B_{1/t_0}(K-N+1, N)
\]

where \( B_g(h, l) \) is the incomplete beta function defined as

\[
B_g(h, l) = \int_0^g t^{h-1} (1 - t)^{l-1} \, dt
\]

Probability of detection:

\[
P_D = 1 - \frac{1}{t_0^K} \sum_{j=1}^{K-N+1} \binom{K}{N+j-1} (t_0 - 1)^{N+j-1} \frac{G_j(\lambda_0/t_0)}{n!}
\]

\[
\lambda_0 = y_s^H R_n^{-1} y_s = \text{SNR}
\]

\[
G_j(t) = e^{-t} \sum_{n=0}^{j-1} \frac{t^n}{n!}
\]
Let $\mathbb{C}^N$ denote the vector space of all complex N-vectors. Let signal $y_s$ belong to known M-dimensional subspace $V_1$.

$$V_1 = \text{span of all complex } N\text{-vectors whose last } N - M \text{ components are zero.}$$

Projection of a given N-vector $y$ onto subspace $V_1$ is $[y_1^T \ 0_{N-M}^T]^T$

$y_1$ is a vector of M components

$0_{N-M}$ is the zero vector of N-M components

Let $V_2$ denote the orthogonal complement of $V_1$ in $V$.

Projection of $y$ onto subspace $V_2$ is $[0_M^T \ y_2^T]^T$

$y_2$ is a vector of N-M components.

Every N-vector $y$ can be represented as $y = [y_1^T \ y_2^T]^T$

Every signal vector $y_s$ can be written as $[y_{s1}^T \ 0_{N-M}^T]^T$
CFAR Detection of Unknown Signal in Known subspace

\[
\text{GLR} = \frac{\text{Maximized likelihood function under } H_1}{\text{Maximized likelihood function under } H_0}
\]

Maximization is subject to the constraint \( \hat{y}_s \) belongs to \( V_1 \)

\[
\text{GLR} = \left[ 1 + \frac{y_{1,2}^H S_{1,2}^{-1} y_{1,2}}{1 + y_2^H S_{22}^{-1} y_2} \right]^{k+1}
\]

Where

\[
y_{1,2} = y_1 - S_{12} S_{22}^{-1} y_2
\]

\[
S_{1,2} = S_{11} - S_{12} S_{22}^{-1} S_{21}
\]

\[
S_{mn} = \sum_{j=1}^{k} y_m(j) y_n^H(j)
\]

GLRT:

\[
\gamma = \frac{y_{1,2}^H S_{1,2}^{-1} y_{1,2}}{1 + y_2^H S_{22}^{-1} y_2} > (t_0 - 1) \quad < \quad \text{ }(t_0 - 1)
\]
CFAR Detection of Unknown Signal in Known subspace

Probability of False Alarm :

\[ P_{FA} = 1 - \frac{1}{t_0^{K-N+M}} \sum_{j=1}^{K-N+M} \left( \frac{K-N+M}{M+j-1} \right) (t_0 - 1)^{M+j-1} \]

Probability of Detection :

\[ P_D = 1 - \frac{1}{t_0^{K-N+M}} \sum_{j=1}^{K-N+M} \left( \frac{K-N+M}{M+j-1} \right) (t_0 - 1)^{M+j-1} D_j(x/t_0) \]

\[ D_j(x) = \int_0^1 G_j(xr) f_R(r) dr \]

\[ f_R(r) = \frac{K!}{(K-N+M)!(N-M-1)!} r^{K-N+M} (1-r)^{N-M-1}, \quad 0 \leq r \leq 1 \]

\[ G_j(t) = e^{-t} \sum_{n=0}^{j-1} \frac{t^n}{n!} \]
DETECTION OF SOURCE AT UNKNOWN LOCATION IN OCEAN

\[ x = \mathbf{Hq} + \mathbf{w} \]

Signal vector \( \mathbf{Hq} \) belongs to \( \mathcal{M} \neq V_1 \)

Transform \( x \) so that \( \mathbf{Hq} \) is transformed into a vector in \( V_1 \).
Unitary Transformations

**Objective:** Transform signal vector $Hq$ into desired form. Consider the singular value decomposition (SVD)

$$H = UDV^H$$

$U$ is a $N \times N$ unitary matrix whose columns are the eigenvectors of $HH^H$

$V$ is a $M \times M$ unitary matrix whose columns are the eigenvectors of $H^HH$

$D$ is a $N \times M$ diagonal matrix with real non-negative diagonal elements; $N > M$

Define the transformations:

Data vector: $y = U^Hx$

Noise vector: $n = U^Hw$

Signal vector: $y_s = U^Hq = DV^Hq = \begin{bmatrix} y_{s1}^T & 0_{N-M}^T \end{bmatrix}^T$

Secondary data vectors: $y(j) = U^Hx(j); j = 1,...,K$
Modified Detection Problem

\[ H_0 : \quad y = n \]
\[ H_1 : \quad y = y_s + n \]

• \( y_s \) is of the form \( [y_{s1}^T \quad o_{N-M}^T]^T \)

Subspace detection method is applicable to this modified detection problem.
Simulation Results

Arrays with $N = 6$ sensors and inter-sensor spacing $d = 9.6$ m. Sensor depths are $\{z_n = 15.0\text{m, ..., 63.0m}\}$ for the vertical array.

Pekeris model of ocean
Water: depth $h = 70$ m
  density $\rho = 1000\text{kg/m}^3$
  sound speed $c = 1500\text{m/s}$
Bottom: density $\rho_b = 1500\text{kg/m}^3$
  sound speed $c_b = 1700\text{m/s}$
Source frequency = 78 Hz
Range $r = 5000$ m, depth $z_s = 40$ m, bearing $\theta = 45^\circ$. 
Probability of detection Vs. SNR

![Graph showing Probability of detection Vs. SNR](image)

- **UD-SS**
- **SD-SS**
- **SD-AVS**
- **UD-AVS**

Signal-to-Noise Ratio (dB) vs. Probability of Detection for different detection methods.
Receiver Operating Characteristics

SNR = -2 dB

Probability of Detection

Probability of False Alarm
Effect of signal subspace dimension on subspace detector

(a) Probability of False Alarm $= 0.1$

(b) Probability of False Alarm $= 0.1$

- Signal-to-Noise Ratio (dB)
- Probability of Detection

Graph shows the relationship between signal-to-noise ratio and probability of detection for different numbers of modes ($M = 1, 2, 3, 5, 6$).
Bearing Estimation in Shallow Ocean

Subspace Intersection Method

Range-independent ocean with M normal modes
- J sources with distinct bearing angles
- Uniform horizontal linear array of N sensors

\[ N \geq M(J+1) \]

Array signal vector \( \mathbf{s} \) belongs to \( J \)-dimensional signal subspace \( \mathcal{S} \)

\[ \mathcal{S} = \text{span} \{ \mathbf{u}_1, \cdots, \mathbf{u}_J \} \]

\( \{ \mathbf{u}_1, \cdots, \mathbf{u}_J \} \) are eigenvectors corresponding to \( J \) largest eigenvalues of array data correlation matrix \( \mathbf{R} = \mathbb{E}[\mathbf{x}\mathbf{x}^H] \)
For a source at bearing $\theta_i$, array signal vector $s$ also belongs to $M$-dimensional modal subspace $\mathcal{M}(\theta_i)$

$$\mathcal{M}(\theta_i) = \text{span} \{h_1(\theta_i), \ldots, h_M(\theta_i)\}$$

$$h_m(\theta_i) = \text{m}^{th} \text{ column of } H(\theta_i); m = 1, \ldots, M$$

Signal subspace $\mathcal{I}$ and modal subspace $\mathcal{M}(\theta)$ intersect if and only if $\theta \in \{\theta_1, \theta_2, \ldots \theta_J\}$

Equivalently, $\theta_1, \ldots, \theta_J$ are the locations of the $J$ largest peaks of SIM ambiguity function

$$B_{SIM}(\theta) = \left[ \min_{M+1 \leq i \leq M+J} |t_{ii}(\theta)|^{-1} \right]$$
$t_{ii}(\theta)$ is the $i^{th}$ diagonal element of matrix $\mathbf{H}(\theta)$ which is defined as follows

Define $\mathbf{D}(\theta) = [\bar{\mathbf{h}}_1(\theta) \ldots \bar{\mathbf{h}}_M(\theta) \mathbf{u}_1 \ldots \mathbf{u}_J]$ where $\bar{\mathbf{h}}_m(\theta)$ is the normalized version of $\mathbf{h}_m(\theta)$

Perform Cholesky decomposition of $\mathbf{D}(\theta)$

$$\mathbf{D}(\theta) = \mathbf{Q}(\theta) \mathbf{T}(\theta)$$

where columns of $\mathbf{Q}(\theta)$ are orthonormal vectors and $\mathbf{T}(\theta)$ is an upper triangular matrix.
\( \mathbf{R} = E[\mathbf{x} \mathbf{x}^H] \) is estimated from \( L (<\infty) \) snapshots of array data vector \( \mathbf{x} \).

\[
\hat{\mathbf{R}} = \frac{1}{L} \sum_{i=1}^{L} \mathbf{x}(i)\mathbf{x}(i)^H
\]

\( \hat{\mathbf{R}} \rightarrow \mathbf{R} \) as \( L \rightarrow \infty \) in m. s. sense

\( \hat{\theta}_j \rightarrow \theta_j \) as \( L \rightarrow \infty \), for all \( j \)
Fig. 2. SIM response function for AVS array with N=10. Three sources at $30^\circ, 35^\circ, 120^\circ$. SNR=15.23dB for each source, $r = 5000$ m, $f = 50$Hz, $L = 350$. 
Fig. 3. SIM response function for scalar sensor array with N=10. Three sources at $30^\circ, 35^\circ, 120^\circ$. SNR=15.23dB for each source, $r = 5000$ m, $f = 50$Hz, $L = 350$. 
Fig. 5. RMSE vs SNR for AVS and scalar sensor arrays. One source at $60^0$, $r = 5000$ m, $f = 50$Hz, $L = 350$. 
Fig. 9. RMSE vs bearing angle for AVS and scalar sensor arrays. SNR = 20dB, 
r = 5000 m, f = 50Hz, L = 350.
Fig. 15. RMSE vs $d$ for a source at $60^0$. AVS and scalar sensor arrays with $N=10$, $f = 50$Hz, $SNR = 20$dB, $r = 5000$m, $L = 350$. Wavelength $\lambda = 30$m.
Fig. 16. Probability of resolution vs $d$ for two sources at $60^0$ and $62^0$. AVS and scalar sensor arrays with $N=20$, $f = 50$Hz, SNR = 20dB, $r = 5000$m, $L = 350$. Wavelength $\lambda = 30$m.
Conclusions

• Subspace-based techniques of signal detection and acoustic source bearing estimation in shallow ocean have been presented.

• Performance of algorithms based on vector sensor array is significantly better than that of corresponding algorithms based on conventional hydrophone array.